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On the Drinfel'd-Kohno Equivalence of Groups and Quantum Groups

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Abstract

A method to calculate matrix representations of the twist element $\mathcal{F}$ of Drinfel'd – chosen to be unitary – is given and illustrated at some examples. It is observed that for these F-matrices the crystal limit $q \to 0$ exists and that F-matrices twisting from 0 to q are of a simpler form than F-matrices twisting from 1 to q. These results lead to a new interpretation of q-deformation in terms of tensor products of finite-dimensional representations of compact simple Lie groups.

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1 Introduction

Drinfel'd published his ground breaking work on deformation of semi-simple Lie algebras in 1989 in Russian and in 1990 in English [1]. Combining ideas from areas of physics and mathematics as diverse as Lie theory, Hopf algebra theory, integrable models, conformal field theory, gauge theory, cohomology theory and category theory he proved their existence and uniqueness using the mathematical frame of *quasitriangular quasi-Hopf quantized universal enveloping algebras* and *twists* as equivalence mappings between them. Interestingly he solved a *mathematical* problem by translating it into a *physical* model, thereby somehow interchanging the roles usually played by the two sciences.

Impressive as it is the Drinfel'd-Kohno theorem has been suffering from one crucial drawback: Even though the existence of the twist between the "undeformed" and the "deformed" universal enveloping algebras is proven any attempt to gain an explicit expression for the twist element $\mathcal{F}$ failed. It is mostly due to this circumstance that the theorem didn't have a major impact on research papers published ever since. Its content remained inaccesible for purposes of explicit calculations.

The present work is an attempt to overcome these difficulties.

2 The Drinfel'd-Kohno Theorem

Theorem (DRINFEL'D-KOHNO,[1]) *Let* $\mathbf{g}$ *be a simple finite-dimensional Lie algebra over* $\mathbb{C}$ *and* $\boldsymbol{t}$ *a symmetric element of* $\mathbf{g}\otimes\mathbf{g}$ *s.t.* $[\boldsymbol{t}, \Delta(\mathbf{g})] = 0$*. From this can be constructed, on the one hand, a quasitriangular quasi-Hopf quantized universal enveloping algebra (QtQHQUEA)* $(U\mathbf{g}[[h]], \Delta, \Phi, \mathcal{R})$*, where* Δ *is the undeformed cocommutative and coassociative coproduct and* $\mathcal{R} = \exp(\frac{h}{2}\boldsymbol{t})$*, and, on the other hand, a quasitriangular Hopf quantized universal enveloping algebra (QtHQUEA)* $(U_h\mathbf{g}, \bar{\Delta}, \bar{\mathcal{R}})$*. They are twist-equivalent as quasitriangular quasi-Hopf algebras (QtQHAs).*

That is there exists an invertible $\mathcal{F} \in U\mathbf{g}^{\otimes 2}[[h]]$ *s.t.* *

$$\bar{\Delta}(x) = \mathcal{F}\,\Delta(x)\,\mathcal{F}^{-1} \quad \forall x \in U\mathbf{g}[[h]] \tag{1}$$

$$\bar{\mathcal{R}}_{12} = \mathcal{F}_{21}\,\mathcal{R}_{12}\,\mathcal{F}_{12}^{-1} \tag{2}$$

$$\mathbb{1}^{\otimes 3} = \bar{\Phi} = \mathcal{F}_{23}\;(\mathrm{id}\otimes\Delta)(\mathcal{F})\;\Phi\;[\mathcal{F}_{12}\;(\Delta\otimes\mathrm{id})(\mathcal{F})]^{-1}. \tag{3}$$

To begin with we discuss some implications:

The theorem establishes an *invertible* transition from the 'undeformed' to the 'deformed' case without refering to the limit $h \to 0$. This is in clear contrast to the relationship between classical and quantum physics: As elements of a commutative algebra phase space functions have completely 'forgotten' about the possibility of being non-commutative. To restore this information it takes an additional structure, the Poisson bracket, which is independent of the commutative multiplication. To put it another way: The commutative algebra of classical phase space functions *without* Poisson structure contains strictly less information than the non-commutative algebra of operators on the Hilbert space of states.

In the case of (commutative) function algebras on groups and non-commutative 'function algebras on quantum groups' (which are the respective dual Hopf algebras of the universal enveloping algebras featured in the theorem) the two different algebra structures are related by some kind of conjugation (or, more precisely, the dual version of it, 'co-conjugation') or 'gauge transformation' (Drinfel'd) which is, and that is the crucial point, invertible. This implies that both structures contain the *same* information.

Whereas we think of (non-commuting) operators and (commuting) phase space functions as very different kinds of objects the Drinfel'd-Kohno theorem strongly suggests to think of (non-commuting) 'functions on a quantum group' and (commuting) functions on a group as the same kind of objects, and, in particular, *not as primarily linked by the limit* $h \to 0$ – which is not to say that they are not *additionally* linked by that limit. Their relationship is very similiar to that between a covariant derivative and an ordinary one. Although one might think of the latter as the former in the limit of vanishing coupling constant this is not a very natural way to look at it.

*For $x = a \otimes b$ let $x_{12} = a \otimes b$, $x_{21} = b \otimes a$ resp. $x_{12} = a \otimes b \otimes \mathbb{1}$, $x_{23} = \mathbb{1} \otimes a \otimes b$, etc.

3 F-matrices

In addition to what is stated in the Drinfel'd-Kohno theorem we will use the following input to explicitly calculate matrix representations of $\mathcal{F}$:

1. The matrix representations of $\mathcal{R}$ and $\bar{\mathcal{R}}$ can be calculated.[†]
2. $\mathcal{F}$ commutes with the coproducts of the generators of the Cartan subalgebra of $\mathbf{g}$.
3. The quantized universal enveloping algebras of compact semi-simple Lie groups are for real h equipped with a $*$-structure such that the matrix representations are $*$-representations.
4. $\lim\limits_{h\to 0} \mathcal{F} = \mathbb{1}^{\otimes 2}$

We will use the following notation: Lower case Greek letters label matrices as a whole, and if the same Greek label appears repeatedly in one expression we understand this as ordinary matrix multiplication. Thus, $a_\alpha b_\alpha$ means the same as $a^i_j b^j_k$, and $a_\alpha b_\beta c_\alpha d_\beta$ means the same as $a^i_j b^k_l c^j_m d^l_n$. If a symbol has several Greek labels each of them stands for a pair of an upper and a lower index: $a_{\alpha\beta} b_{\alpha\gamma} c_\beta$ means the same as $a^{ik}_{jl} b^{jp}_{mq} c^l_n$. Symbols without Greek labels will not have any matrix indices. This allows us to denote matrix representations of (quasi-)Hopf algebra elements by the same symbol as the (quasi-)Hopf algebra elements themselves with Greek labels attached to them: $x_\alpha = \varrho_\alpha(x), \quad \mathcal{R}_{\alpha\beta} = (\varrho_\alpha \otimes \varrho_\beta)(\mathcal{R})$. Since only coinciding labels indicate index contraction, different labels may refer to different matrix representations (of possibly different dimensions).

3.1 Unitarity

From $\mathcal{R}^*_{12} = \mathcal{R}_{12} = \mathcal{R}_{21}$ and $\bar{\mathcal{R}}^*_{12} = \bar{\mathcal{R}}_{21}$ we deduce the identity

$$\mathcal{F}^{-1*}_{12}\,\mathcal{R}_{12}\,\mathcal{F}^*_{21} = \mathcal{F}^{-1*}_{12}\,\mathcal{R}^*_{12}\,\mathcal{F}^*_{21} = \bar{\mathcal{R}}^*_{12} = \bar{\mathcal{R}}_{21} = \mathcal{F}_{12}\,\mathcal{R}_{12}\,\mathcal{F}^{-1}_{21}, \tag{4}$$

thus

$$\mathcal{F}^*\,\mathcal{F}\,\mathcal{R}^2 = \mathcal{R}^2\,\mathcal{F}^*\,\mathcal{F}.$$

[†]The latter ones follow from the defining representations given in [2] for the series A, B, C, D.

From $\Delta(x^*) = \Delta(x)^*$ and $\bar{\Delta}(x^*) = \bar{\Delta}(x)^*$ it follows that

$$\mathcal{F}^* \mathcal{F} \Delta(x) = \Delta(x) \mathcal{F}^* \mathcal{F}.$$

Thus twisting by $\mathcal{F}^* \mathcal{F}$ leaves the undeformed universal enveloping algebra invariant such that twisting by the unitary

$$\tilde{\mathcal{F}} = \mathcal{F}(\mathcal{F}^* \mathcal{F})^{-\frac{1}{2}}$$

gives the same result as twisting by $\mathcal{F}$. This has been noted by Jurčo [3]. We will therefore assume from here without loss of generality that $\mathcal{F}$ is unitary: $\mathcal{F}^* = \mathcal{F}^{-1}$.

Since all $\mathcal{R}_{\alpha\beta}$ and $\bar{\mathcal{R}}_{\alpha\beta}$ are real matrices the corresponding $\mathcal{F}_{\alpha\beta}$ are real, orthogonal matrices:

$$\mathcal{F}_{\alpha\beta}^{\top} = \mathcal{F}_{\alpha\beta}^{-1}. \tag{5}$$

3.2 Orthogonal Projectors

Denote by H_i, ($i = 1, \ldots,$ rank $\mathbf{g}$), the generators of the Cartan subalgebra of $\mathbf{g}$. In the representation $\varrho_\alpha \otimes \varrho_\beta$ we write down their expansion in terms of eigenvalues and orthonormal projectors.

$$\Delta(H_i)_{\alpha\beta} = (H_i \otimes \mathbb{1} + \mathbb{1} \otimes H_i)_{\alpha\beta} = \sum_c \eta_{i,c} P_{\alpha\beta}^{\langle i,c\rangle}$$

$$P_{\alpha\beta}^{\langle i,c\rangle} P_{\alpha\beta}^{\langle i,c'\rangle} = \delta_{cc'} P_{\alpha\beta}^{\langle i,c\rangle}$$

From $\Delta(H_i)\mathcal{F} = \mathcal{F}\Delta(H_i)$ it follows that

$$P_{\alpha\beta}^{\langle i,c\rangle} \mathcal{F}_{\alpha\beta} = \mathcal{F}_{\alpha\beta} P_{\alpha\beta}^{\langle i,c\rangle} \quad \forall i, c \tag{6}$$

Now set $\mathcal{Q} = \mathcal{R}^2 = \mathrm{e}^{h\boldsymbol{t}}$ and $\bar{\mathcal{Q}} = \bar{\mathcal{R}}_{21}\bar{\mathcal{R}}_{12}$,[‡] such that $\bar{\mathcal{Q}}\mathcal{F} = \mathcal{F}\mathcal{Q}$. Then, analogously, we obtain for $\mathcal{Q}$ and $\bar{\mathcal{Q}}$:

$$\mathcal{Q}_{\alpha\beta} = \sum_k \lambda_k \, P^{(k)}_{\alpha\beta}$$

$$\bar{\mathcal{Q}}_{\alpha\beta} = \sum_k \lambda_k \bar{P}^{(k)}_{\alpha\beta}$$

with

$$P^{(k)}_{\alpha\beta} P^{(\ell)}_{\alpha\beta} = \delta_{k\ell} \; P^{(k)}_{\alpha\beta}.$$

$$\bar{P}^{(k)}_{\alpha\beta} \; \bar{P}^{(\ell)}_{\alpha\beta} = \delta_{k\ell} \; \bar{P}^{(k)}_{\alpha\beta}.$$

From $\bar{\mathcal{Q}}_{\alpha\beta}\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\alpha\beta}\mathcal{Q}_{\alpha\beta}$ we have

$$\bar{P}^{(k)}_{\alpha\beta}\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\alpha\beta}P^{(k)}_{\alpha\beta} \tag{7}$$

leading to

$$\prod_{i=1}^{\text{rank}\,\mathbf{g}} P^{\langle i,c_i\rangle}_{\alpha\beta} \, \bar{P}^{(k)}_{\alpha\beta}\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\alpha\beta} \prod_{i=1}^{\text{rank}\,\mathbf{g}} P^{\langle i,c_i\rangle}_{\alpha\beta} P^{(k)}_{\alpha\beta} \tag{8}$$

This identity determines the orthogonal $\mathcal{F}_{\alpha\beta}$ uniquely whenever there are complete sets of one-dimensional projectors $P^a_{\alpha\beta}$ and $\bar{P}^a_{\alpha\beta}$ among the $\prod_{i=1}^{\text{rank}\,\mathbf{g}} P^{\langle i,c_i\rangle}_{\alpha\beta} P^{(k)}_{\alpha\beta}$ and $\prod_{i=1}^{\text{rank}\,\mathbf{g}} P^{\langle i,c_i\rangle}_{\alpha\beta} \bar{P}^{(k)}_{\alpha\beta}$, respectively. (The limit $\lim_{h\to 0} \mathcal{F} = \mathbb{1}^{\otimes 2}$ fixes all signs uniquely.) This then gives us natural bases of the representation space of $\varrho_\alpha \otimes \varrho_\beta$ for general h and for $h=0$. Switching to the parameter $q = \mathrm{e}^h$ for later convenience, we denote the basis vectors by $|a;q\rangle$ and $|a;1\rangle$ with orthogonality and completeness relations

$$\langle a;q|a';q\rangle = \delta_{aa'}$$

[‡]We use representations of $\bar{\mathcal{R}}_{21}\bar{\mathcal{R}}_{12}$, rather than the conventional $\hat{R}$ which only works if $\varrho_\alpha = \varrho_\beta$ (in which case $Q = \hat{R}^2$).

$$\sum_a |a;q\rangle\langle a;q| = \mathbb{1}_\alpha \mathbb{1}_\beta,$$

and

$$P^a_{\alpha\beta} = |a;1\rangle\langle a;1|,$$

$$\bar{P}^a_{\alpha\beta} = |a;q\rangle\langle a;q|.$$

Thus we finally obtain

$$\mathcal{F}_{\alpha\beta} = \sum_a |a;q\rangle\langle a;1|. \tag{9}$$

From eq.(3) it follows that

$$\Phi = [\mathcal{F}_{23}(\mathrm{id}\otimes\Delta)(\mathcal{F})]^{-1}\mathcal{F}_{12}(\Delta\otimes\mathrm{id})(\mathcal{F}) \tag{10}$$

or

$$\Phi_{\alpha\beta\gamma} = [\mathcal{F}_{\beta\gamma}(\mathrm{id}\otimes\Delta)(\mathcal{F})_{\alpha\beta\gamma}]^{-1}\mathcal{F}_{\alpha\beta}(\Delta\otimes\mathrm{id})(\mathcal{F})_{\alpha\beta\gamma}. \tag{11}$$

4 Examples for F-matrices

4.1 su(n)

Let now $\mathbf{g} = \mathrm{su}(n)$, ϱ_α and ϱ_β two copies of the fundamental, n-dimensional representation and e_i $(i = 1, \ldots, n)$ its orthonormal standard basis with dual e^i. In this case we find that we indeed obtain a complete set of one-dimensional projectors by the method introduced above. For general (real, positive) q we find the following basis in terms of the product basis $e_i \otimes e_j$:

$$\begin{aligned} &\text{for } i = j: \quad |ii;q\rangle &&= e_i \otimes e_i \\ &\text{for } i < j: \quad |ij_+;q\rangle &&= \tfrac{1}{\sqrt{q+q^{-1}}} \left(\sqrt{q}\, e_i \otimes e_j + \sqrt{q^{-1}}\, e_j \otimes e_i\right) \\ &\text{for } i < j: \quad |ij_-;q\rangle &&= \tfrac{1}{\sqrt{q+q^{-1}}} \left(\sqrt{q^{-1}}\, e_i \otimes e_j - \sqrt{q}\, e_j \otimes e_i\right) \end{aligned} \tag{12}$$

That implies

$$\begin{aligned} &\text{for } i = j: \quad \langle ii;1| &&= e^i \otimes e^i \\ &\text{for } i < j: \quad \langle ij_+;1| &&= \tfrac{1}{\sqrt{2}}(e^i \otimes e^j + e^j \otimes e^i) \\ &\text{for } i < j: \quad \langle ij_-;1| &&= \tfrac{1}{\sqrt{2}}(e^i \otimes e^j - e^j \otimes e^i) \end{aligned} \tag{13}$$

and, finally,[§]

$$\begin{aligned} F = \; & \sum_i (e_i \otimes e_i)(e^i \otimes e^i) \\ & + \sum_{i<j} \tfrac{1}{\sqrt{2(q+q^{-1})}} \left[\left(\sqrt{q}\, e_i \otimes e_j + \sqrt{q^{-1}}\, e_j \otimes e_i\right)(e^i \otimes e^j + e^j \otimes e^i) \right. \\ & \left. + \left(\sqrt{q^{-1}}\, e_i \otimes e_j - \sqrt{q}\, e_j \otimes e_i\right)(e^i \otimes e^j - e^j \otimes e^i) \right]. \end{aligned} \tag{14}$$

For general representations of su(n) we have one projector for every Young diagram. As far as the eigenvalues of the Cartan generators uniquely fix a one-dimensional subspace of an irreducible representation it is guaranteed that our procedure will yield a unique F-matrix for an arbitrary pair of finite-dimensional representations $(\varrho_\alpha, \varrho_\beta)$ of su(n). If there are remaining degeneracies one might have to look for sufficiently natural additional input to fix a unique basis, as will be done below in the case of the defining representations of the Lie algebras of the series B, C, D.

[§] We write F for $\mathcal{F}_{\alpha\beta}$ whenever we give an explicit matrix to emphasize that relations containing $\mathcal{F}_{\alpha\beta}$ are valid for general ϱ_α and ϱ_β.

4.2 so(n) and sp(n)

If we try to apply our method to a pair of fundamental representations of so(n) or sp(n) it turns out that there are still some degeneracies. A closer look at the eigenspaces of the Cartan generators reveals beside several one- and two-dimensional subspaces (the latter decompose in a symmetric and an antisymmetric one, as in the case of su(n)) in this case also an n-dimensional subspace. With the trace projector disposing us of one dimension we are left with $n-1$ dimensions to distribute among the symmetric and antisymmetric projectors. Since this problem is increasing with the dimension of the Lie algebra's defining representation an inductive procedure might help.

For so(3) there is no degeneracy at all. In the case of so(4) we are left with a two-dimensional antisymmetric subspace and in the case of sp(4) with a two-dimensional symmetric one.

If one takes a look at the explicit expressions for the R-matrices of the fundamental representations of the three series B, C, D [2] one notices their characteristic 'onion-like' structure. In the center of such an R-matrix one finds the R-matrices of all the Lie algebras of the same series with lower rank. So one can construct the Lie algebras of higher rank from those of lower rank in the same series by 'adding coordinates at the outside'. This shows the way to a reasonably natural assumption that will fix a unique orthonormal basis of the representation space of the tensor product of two defining representations of the Lie algebras of these series, and thus provide an F-matrix.

We assume that the symmetric and antisymmetric eigenvectors for the Lie algebra of one of these series of rank $r+1$ are the same as for the one of rank r of the same series plus one additional symmetric and antisymmetric one, respectively, which are determined by the orthogonality and normalization requirement.

We denote by n the dimension of the defining representation of a Lie algebra of one of the series B, C, D. For ϱ_α and ϱ_β we take two copies thereof. We additionally use $s = \frac{n}{2}$ (for B and D) resp. $s = \frac{n}{2}+1$ (for C). As before we denote the standard basis vectors of the representation space by e_i and their duals by e^i. Again we express our basis for the representation space of the tensor product of the two representations in terms of the product basis $e_i \otimes e_j$.

In the case $i+j \neq n+1$ we find the corresponding basis vectors of the same form as eq.(12). Using $\bar{\imath} = n+1-i$ and $\{k\} = q^k - q^{-k}$ we obtain by the inductive

construction in the remaining cases the following basis vectors.[¶] In terms of these the F-matrices are then given by eq.(9).

D_s:

$$
\begin{aligned}
&\text{for } k = 1,\ldots,s-1: \\
|n_{+(k)};q\rangle \;&=\; \sqrt{\tfrac{\{1\}^3}{\{2\}\{k\}\{k+1\}}}\Big[\tfrac{\{k\}}{\{1\}}(q\, e_{s-k}\otimes e_{\overline{s-k}} + q^{-1}\, e_{\overline{s-k}}\otimes e_{s-k}) \\
&\qquad - \sum_{i=0}^{k-1}(q^{-i}\, e_{s-i}\otimes e_{\overline{s-\imath}} + q^{i}\, e_{\overline{s-\imath}}\otimes e_{s-i})\Big] \\
\langle n_{+(k)};1| \;&=\; \sqrt{\tfrac{1}{2k(k+1)}}\Big[k(e^{s-k}\otimes e^{\overline{s-k}} + e^{\overline{s-k}}\otimes e^{s-k}) \\
&\qquad - \sum_{i=0}^{k-1}(e^{s-i}\otimes e^{\overline{s-\imath}} + e^{\overline{s-\imath}}\otimes e^{s-i})\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\text{for } k = 0,\ldots,s-1: \\
|n_{-(k)};q\rangle \;&=\; \sqrt{\tfrac{\{1\}^3\{k-1\}\{k\}}{\{2\}\{2k-2\}\{2k\}}}\Big[\tfrac{\{2k-2\}}{\{1\}\{k-1\}}(e_{s-k}\otimes e_{\overline{s-k}} - e_{\overline{s-k}}\otimes e_{s-k}) \\
&\qquad + \sum_{i=0}^{k-1}(q^{-i}\, e_{s-i}\otimes e_{\overline{s-\imath}} + q^{i}\, e_{\overline{s-\imath}}\otimes e_{s-i})\Big] \\
\langle n_{-(k)};1| \;&=\; \sqrt{\tfrac{1}{2}}(e^{s-k}\otimes e^{\overline{s-k}} - e^{\overline{s-k}}\otimes e^{s-k})
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&\text{and} \\
|n_{\mathrm{Tr}};q\rangle \;&=\; \sqrt{\tfrac{\{1\}\{s-1\}}{\{s\}\{2s-2\}}}\sum_{i=0}^{s-1}(q^{-i}\, e_{s-i}\otimes e_{\overline{s-\imath}} + q^{i}\, e_{\overline{s-\imath}}\otimes e_{s-i}) \\
\langle n_{\mathrm{Tr}};1| \;&=\; \sqrt{\tfrac{1}{2s}}\sum_{j=0}^{2s} e^{j}\otimes e^{\bar{\jmath}}
\end{aligned}
$$

[¶] Thus for $q=1$ the expressions are to be replaced by their respective limits. Note that $\lim_{q\to 1}\frac{\{k\}}{\{m\}} = \frac{k}{m}$ and $\lim_{q\to 1}\{k\} = 0$.

$B_{s-\frac{1}{2}}$:

$$
\begin{aligned}
&\text{for } k = \tfrac{1}{2}, \tfrac{3}{2}, \ldots, s-1 : \\
|n_{+(k)}; q\rangle \;&=\; \sqrt{\tfrac{\{1\}^3}{\{2\}\{k\}\{k+1\}}}\Big[\tfrac{\{k\}}{\{1\}}(q\, e_{s-k} \otimes e_{\overline{s-k}} + q^{-1}\, e_{\overline{s-k}} \otimes e_{s-k}) \\
&\quad -e_{s+\frac{1}{2}} \otimes e_{s+\frac{1}{2}} - \sum_{i=\frac{1}{2}}^{k-1}(q^{-i}\, e_{s-i} \otimes e_{\overline{s-\imath}} + q^{i}\, e_{\overline{s-\imath}} \otimes e_{s-i})\Big] \\
\langle n_{+(k)}; 1| \;&=\; \sqrt{\tfrac{1}{2k(k+1)}}\Big[k(e^{s-k} \otimes e^{\overline{s-k}} + e^{\overline{s-k}} \otimes e^{s-k}) \\
&\quad -e^{s+\frac{1}{2}} \otimes e^{s+\frac{1}{2}} - \sum_{i=\frac{1}{2}}^{k-1}(e^{s-i} \otimes e^{\overline{s-\imath}} + e^{\overline{s-\imath}} \otimes e^{s-i})\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\text{for } k = \tfrac{1}{2}, \tfrac{3}{2}, \ldots, s-1 : \\
|n_{-(k)}; q\rangle \;&=\; \sqrt{\tfrac{\{1\}^3\{k-1\}\{k\}}{\{2\}\{2k-2\}\{2k\}}}\Big[\tfrac{\{2k-2\}}{\{k-1\}\{1\}}(e_{s-k} \otimes e_{\overline{s-k}} - e_{\overline{s-k}} \otimes e_{s-k}) \\
&\quad +e_{s+\frac{1}{2}} \otimes e_{s+\frac{1}{2}} + \sum_{i=\frac{1}{2}}^{k-1}(q^{-i}\, e_{s-i} \otimes e_{\overline{s-\imath}} + q^{i}\, e_{\overline{s-\imath}} \otimes e_{s-i})\Big] \\
\langle n_{-(k)}; 1| \;&=\; \sqrt{\tfrac{1}{2}}(e^{s-k} \otimes e^{\overline{s-k}} - e^{\overline{s-k}} \otimes e^{s-k})
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
|n_{\mathrm{Tr}}; q\rangle \;&=\; \sqrt{\tfrac{\{1\}\{s-1\}}{\{s\}\{2s-2\}}}\Big[e_{s+\frac{1}{2}} \otimes e_{s+\frac{1}{2}} \\
&\quad + \sum_{i=\frac{1}{2}}^{s-1}(q^{-i}\, e_{s-i} \otimes e_{\overline{s-\imath}} + q^{i}\, e_{\overline{s-\imath}} \otimes e_{s-i})\Big] \\
\langle n_{\mathrm{Tr}}; 1| \;&=\; \sqrt{\tfrac{1}{2s}} \sum_{j=0}^{2s} e^{j} \otimes e^{\overline{\jmath}}
\end{aligned}
$$

C_{s-1}:

$$
\begin{aligned}
&\text{for } k=1,\dots,s-1: \\
|n_{+(k)};q\rangle &= \sqrt{\tfrac{\{1\}^3\{k\}\{k+1\}}{\{2\}\{2k\}\{2k+2\}}}\Big[\tfrac{\{2k\}}{\{1\}\{k\}}(q\,e_{s-k}\otimes e_{\overline{s-k}}+q^{-1}\,e_{\overline{s-k}}\otimes e_{s-k}) \\
&\quad +\sum_{i=1}^{k-1}(q^{-i}\,e_{s-i}\otimes e_{\overline{s-i}}-q^{i}\,e_{\overline{s-i}}\otimes e_{s-i})\Big] \\
\langle n_{+(k)};1| &= \sqrt{\tfrac{1}{2}}(e^{s-k}\otimes e^{\overline{s-k}}+e^{\overline{s-k}}\otimes e^{s-k}) \\
\\
&\text{for } k=2,\dots,s-1: \\
|n_{-(k)};q\rangle &= \sqrt{\tfrac{\{1\}^3}{\{2\}\{k-1\}\{k\}}}\Big[\tfrac{\{k-1\}}{\{1\}}(e_{s-k}\otimes e_{\overline{s-k}}-e_{\overline{s-k}}\otimes e_{s-k}) \\
&\quad +\sum_{i=1}^{k-1}(-q^{-i}\,e_{s-i}\otimes e_{\overline{s-i}}+q^{i}\,e_{\overline{s-i}}\otimes e_{s-i})\Big] \\
\langle n_{-(k)};1| &= \sqrt{\tfrac{1}{2(k-1)k}}\Big[(k-1)(e^{s-k}\otimes e^{\overline{s-k}}-e^{\overline{s-k}}\otimes e^{s-k}) \\
&\quad +\sum_{i=1}^{k-1}(-e^{s-i}\otimes e^{\overline{s-i}}+e^{\overline{s-i}}\otimes e^{s-i})\Big]
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
|n_{\mathrm{Tr}};q\rangle &= \sqrt{\tfrac{\{1\}\{s\}}{\{s-1\}\{2s\}}}\sum_{i=1}^{s-1}(q^{-i}\,e_{s-i}\otimes e_{\overline{s-i}}-q^{i}\,e_{\overline{s-i}}\otimes e_{s-i}) \\
\langle n_{\mathrm{Tr}};1| &= \sqrt{\tfrac{1}{2(s-1)}}\sum_{i=1}^{s-1}(e^{s-i}\otimes e^{\overline{s-i}}-e^{\overline{s-i}}\otimes e^{s-i})
\end{aligned}
$$

4.3 The crystal limit $q \to 0$

The explicit expressions for the orthonormal basis vectors for general q, $|a;q\rangle$, now allow the following observation:

For each basis vector $|a;q\rangle$ the limit $q \to 0$ exists and the set of the $|a;0\rangle$ is again an orthonormal basis of the representation space of the respective tensor product of representations.[‖] In terms of the product basis $e_i \otimes e_j$ it turns out to be particularly simple:

$$\begin{aligned}
|ii;0\rangle &= e_i \otimes e_i \\
|ij_+;0\rangle &= e_j \otimes e_i \\
|ij_-;0\rangle &= e_i \otimes e_j \\
|n_{+(k)};0\rangle &= e_{\overline{s-k}} \otimes e_{s-k} \\
|n_{-(k)};0\rangle &= \pm e_{s-k+1} \otimes e_{\overline{s-k+1}} \qquad (+ \text{ for } B \text{ and } D, - \text{ for } C) \\
|n_{\mathrm{Tr}};0\rangle &= e_1 \otimes e_n
\end{aligned} \tag{18}$$

In this limit the tensor product of two standard basis vectors is a basis vector of an irreducible component of the product representation. Thus transition from the basis for $q=1$ to the basis for $q=0$ means transition from a factorized basis to a fully reduced one, and the corresponding F-matrix' entries are therefore the Clebsch-Gordan-coefficients.

This is in agreement with results obtained by Date, Jimbo, Miwa [4] and Kashiwara [5]. Kashiwara calls the limit $q \to 0$ *crystal limit* and the respective bases *crystal bases*.

The existence of $\lim\limits_{q\to 0} \mathcal{F}$ is not ensured by the Drinfel'd-Kohno theorem since in particular for the universal R-matices $\mathcal{R}$ and $\bar{\mathcal{R}}$ the limit does not exist. For unitary $\mathcal{F}$ it is however evident from our construction that its representations survive the limit which furthermore is smooth in every respect.

For two values q, q' we will now write

$$\mathcal{F}^{[q'q]} = \mathcal{F}'\mathcal{F}^{-1}$$

[‖]Since in general $\mathcal{F}_{\alpha\beta}(q^{-1}) = \mathcal{F}_{\beta\alpha}(q)$ one can discuss the limit $q \to \infty$ in full analogy.

such that

$$\mathcal{F}^{[q''q]} = \mathcal{F}^{[q''q']}\mathcal{F}^{[q'q]}$$

$$\mathcal{F}^{[q'q]}_{\alpha\beta} = \sum_a |a;q'\rangle\langle a;q|$$

$$\mathcal{F} = \mathcal{F}^{[q1]}$$

$$\mathcal{F}^{-1} = \mathcal{F}^{[1q]},$$

and, in particular,

$$\mathcal{F}^{[q1]} = \mathcal{F}^{[q0]}\mathcal{F}^{[01]}.$$

The representations of the coboundary of $\mathcal{F}^{[01]}$ (with respect to the "undeformed" coproduct) can then be identified as the Racah coefficients.

$$d\mathcal{F}^{[01]} = \mathcal{F}^{[01]}_{23}(\mathrm{id}\otimes\Delta)(\mathcal{F}^{[01]})(\Delta\otimes\mathrm{id})(\mathcal{F}^{[10]})\mathcal{F}^{[10]}_{12} \tag{19}$$

Note that this is what becomes of the (second) trivial coassociator $\Phi_0 = \mathbb{1}^{\otimes 3}$ of $U\mathbf{g}[[h]]$ under the twist with $\mathcal{F}^{[q1]}$ (cf. eq. 3) in the limit $q\to 0$. It is crucial that this coassociator is *not* compatible with the *quasitriangular* structure considered in the Drinfel'd-Kohno theorem but with the "classical" *triangular* structure given by the universal R-matrices $\mathcal{R}_0=\mathbb{1}^{\otimes 2}$ and $\bar{\mathcal{R}}_0=\mathcal{F}_{21}\mathcal{F}^{-1}_{12}$. In full analogy the expression for Φ given in eq.(10) is the coboundary of $\mathcal{F}^{[q1]-1}=\mathcal{F}^{[1q]}$ with respect to the "deformed" coproduct

$$\bar{d}\mathcal{F}^{[1q]} = \Phi. \tag{20}$$

It now turns out that the matrices $\mathcal{F}^{[q0]}_{\alpha\beta}$ are of much simpler form than the matrices $\mathcal{F}^{[q1]}_{\alpha\beta}$. E.g. for a pair of fundamental representations of su(2) it is

$$F^{[q1]} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{q}+\sqrt{q^{-1}}}{\sqrt{2(q+q^{-1})}} & \frac{\sqrt{q}-\sqrt{q^{-1}}}{\sqrt{2(q+q^{-1})}} & 0 \\ 0 & -\frac{\sqrt{q}-\sqrt{q^{-1}}}{\sqrt{2(q+q^{-1})}} & \frac{\sqrt{q}+\sqrt{q^{-1}}}{\sqrt{2(q+q^{-1})}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sin\varphi+\cos\varphi}{\sqrt{2}} & \frac{\sin\varphi-\cos\varphi}{\sqrt{2}} & 0 \\ 0 & -\frac{\sin\varphi-\cos\varphi}{\sqrt{2}} & \frac{\sin\varphi+\cos\varphi}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}$$

$$F^{[q0]} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{q^{-1}}}{\sqrt{q+q^{-1}}} & \frac{\sqrt{q}}{\sqrt{q+q^{-1}}} & 0 \\ 0 & -\frac{\sqrt{q}}{\sqrt{q+q^{-1}}} & \frac{\sqrt{q^{-1}}}{\sqrt{q+q^{-1}}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

where $q = \tan\varphi$. In higher representations the fact that the form of the $\mathcal{F}^{[q0]}_{\alpha\beta}$ is simpler will be more distinctive.

This is not so surprising if one takes into consideration that the quality of $\mathcal{F}^{[01]}_{\alpha\beta}$ to transform to a fully reduced basis simply means that it diagonalizes $\boldsymbol{t}_{\alpha\beta}$, and therefore $\mathcal{Q}_{\alpha\beta}$. From this point of view it makes good sense that $q=0$ is the natural reference point, rather than $q=1$. This is because it lies in the very nature of tensor products that there is no inherent information that would allow an identification as a "composed" object – as opposed to an "elementary" one. Thus from the perspective of the product representation the factorized basis ($q=1$) is not distinguished in any way from others. The only one for which this is the case is the fully reduced one ($q=0$).

5 Conclusion

Suppose you have two matrix representations of a compact simple Lie group with orthonormal bases chosen such that the Cartan generators are diagonal. There are two natural choices for orthonormal bases for their tensor product, the product basis and the fully reduced basis with respect to which the Cartan generators are still diagonal. Transition between them can be thought of as rotations in the simultaneous eigenspaces of the Cartan generators. This way one can define a one-parameter family of bases, in all of which the Cartan generators are diagonal, assigning the parameter value 0 to the fully reduced basis and the value 1 to the factorized basis such that for every parameter value the basis is orthonormal and the dependence on the parameter is smooth. Then this parameter can be chosen as Drinfel'd's "deformation" parameter $q = \mathrm{e}^h$ and the one-parameter family of bases is given by $|a; q\rangle$.

The Drinfel'd-Kohno theorem shows that quantum groups are equivalent to compact simple Lie groups with an additional structure. It turns out that the additional structure amounts to the information about the degree of factorization of the natural

basis of the tensorproduct of two matrix representations, expressed in terms of the "deformation" parameter q. In view of this it might be more appropriately called a factorization parameter.

Results of this work are applied in [6]. A further application by the author is in progress [7].

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